49th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference
 16t
7 - 10 April 2008, Schaumburg, IL

AIAA 2008-2186

Geometrically Nonlinear First Order Shear Deformation Theory for General Anisotropic Shells

Alberto Pirrera[1] and Paul M. Weaver[2]
University of Bristol, Bristol, UK, BS8 1TR

A generalized first order shear deformation theory for anisotropic multilayered shells is presented. It includes the effects of geometrically nonlinear deformations and general initial curvature. The elasticity equations are expressed in orthogonal curvilinear coordinates lying on the shell's middle surface and hence this formulation turns out to be particularly suitable for the analysis of structures formed using fiber placement manufacturing techniques. A novel expression for the stiffness matrix is presented in which the relationship between the shell shape and the stiffness coefficients is highlighted.

Nomenclature

ξ_1, ξ_2, ζ	=	orthogonal curvilinear coordinates
$\mathbf{r}$	=	position vector of a point on the middle surface
$\mathbf{R}$	=	position vector of an arbitrary point
E, F, G	=	surface metric tensor elements
a_1, a_2	=	scale factors
A_1, A_2	=	Lamé coefficients
R_1, R_2	=	normal radii of curvature of the middle surface
N_{ij}, M_{ij}, Q_{ij}	=	stress resultants per unit length
σ_i	=	stress components
ε_{ij}	=	nonlinear strain components
$\varepsilon_{11}, \varepsilon_{22}$	=	nonlinear elongation of those line elements having, before deformation, directions coincident to the coordinates directions
$\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}$	=	nonlinear shear deformations (change of angles) between those line elements having, before deformation, directions coincident to the coordinates directions
e_{11}, e_{22}	=	linear elongation of those line elements having, before deformation, directions coincident to the coordinates directions
e_{12}, e_{13}, e_{23}	=	linear shears deformations between those line elements having, before deformation, directions coincident to the coordinates directions
$\omega_1, \omega_2, \omega_3$	=	components of the curl of the displacements field
u, v, w	=	displacements
u_0, v_0, w_0	=	displacements of the middle surface of the shell
ϕ_1, ϕ_2	=	rotations of a normal to reference surface
$\overline{Q}_{ij}$	=	transformed stiffnesses, referred to the laminate coordinate directions
K_s	=	shear correction factor
A_{ij}, B_{ij}, D_{ij}	=	stiffness matrix coefficients
$\mathrm{A}_{ij}, \mathrm{B}_{ij}, \Delta_{ij}, \Gamma_{ij}$	=	stiffness matrix coefficients

[1] PhD Student, Dept. of Aerospace Engineering, University of Bristol, UK, AIAA student member
[2] Reader, Dept. of Aerospace Engineering, University of Bristol, UK, AIAA member

E_{ij}, Z_{ij}, H_{ij}	=	stiffness matrix coefficients
δK	=	virtual variation of the kinematic energy
δU	=	virtual variation of the strain energy
δV	=	virtual variation of the potential of the applied forces
I_0, I_1, I_2	=	mass inertias

I. Introduction

THIS work presents a first-order shear deformation theory for multilayered anisotropic thin shells; it is based on the work of Reddy for flat plates.[1] It is the current aim to further develop Reddy's work to shells of general shape also including the effects of geometrically nonlinear deformations as described by Novozhilov.[2]

One of the most remarkable features of composite materials is that they allow engineers to design not only a structure but also its constituent material. Partly due to their excellent specific stiffness, there is the tendency to use them to mimic the well known behavior of isotropic materials.

It is becoming increasingly important for novel applications to exploit the capabilities that composite laminates offer by either increasing structural efficiency or by creating novel functionality. For instance, parts made from unsymmetric stacking sequences have been used rarely because they may introduce several structural couplings and because on cooling-down from cure to room temperature they develop internal stresses and warp. Nonetheless, these phenomena offer great capabilities for novel concepts to be used in emerging research fields like 'elastic tailoring' and 'morphing structures'.

In order to exploit these capabilities it is crucially important to fully understand the structural behavior of the materials and to examine all the sources of anisotropy. The aim of this paper is to gather the understanding to design materials to obtain tailored structural responses of general shells.

For all these reasons the current work attempts to develop a novel model describing shell-like two-dimensional structures. Particular attention has been given to the relationship between curvatures and stiffness coefficients.

Shell structures have been widely used in engineering applications. The literature offers a variety of theories modeling both general elasticity problems and particular design purposes. Each theory or analysis has been developed starting from a common point, namely the indefinite equations of elastic equilibrium. However, they may differ greatly depending on the different purpose-driven assumptions and approximations used. Furthermore, despite the availability of a huge variety of papers dedicated to the study of most shell related structural phenomena, literature almost exclusively applies to the analysis of shell of practical and common use in engineering. Therefore, most published work has been concerned specifically with standard shapes such as cylinders, spheres, cones or generally with shells with small thickness to radius of curvature ratio. As a matter of fact, under this hypothesis, the effect of the curvature on the stiffnesses is often negligible.[1-25]

The present work deals with a generalized first order shear deformation theory for anisotropic multilayered shells. In an attempt to be as general as possible, the model takes into account full anisotropy, general shell geometry, nonlinear deformation and transverse shear deformation.

The elasticity equations are expressed in orthogonal curvilinear coordinates lying on the shell's middle surface. A novel expression for the stiffness matrix is presented. It is also shown that many of the coupling terms are strongly dependent on the shape of the structure.

II. Theoretical Development

In the following sections the theoretical development leading from the governing field equations to the analytical solution, namely the load-displacement equations for shell structures, are presented.

Usual assumptions are followed:

1) Linear elastic behavior of the material.
2) The transverse normal fibers are not elongated.
3) The thickness direction normal stress is negligible compared to other stresses in the same direction.
4) The Love-Kirchhoff hypothesis is relaxed, so those fibers which were straight and normal to the middle plane before deformation remain straight but no longer normal to that plane after deformation.

A. Geometry of Curved Surfaces

It is assumed that the middle surface of the shell structure is described by the orthogonal curvilinear coordinate system (ξ_1, ξ_2, ζ),[1] where ξ_1 and ξ_2 are coordinates describing the position on the middle surface and ζ is the coordinate in the thickness direction. This being the case, points on the middle surface and on an arbitrary position are described respectively by a vector $\mathbf{r} = (\xi_1, \xi_2, 0)$ and $\mathbf{R} = (\xi_1, \xi_2, \zeta)$. The metric properties of a surface are completely described by the first fundamental form. It determines the length of an element of middle surface as

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = E d\xi_1^2 + F d\xi_1 d\xi_2 + G d\xi_2^2 \tag{1}$$

The coefficients represent the elements of the surface metric tensor and are defined as

$$E = \frac{\partial \mathbf{r}}{\partial \xi_1} \cdot \frac{\partial \mathbf{r}}{\partial \xi_1}, \quad F = \frac{\partial \mathbf{r}}{\partial \xi_1} \cdot \frac{\partial \mathbf{r}}{\partial \xi_2}, \quad G = \frac{\partial \mathbf{r}}{\partial \xi_2} \cdot \frac{\partial \mathbf{r}}{\partial \xi_2} \tag{2}$$

In curvilinear coordinates systems the quantities $a_1 = \sqrt{E}$ and $a_2 = \sqrt{F}$ are called scale factors and F has to be identically equal to zero. Similarly, A_1 and A_2, the so called Lamé coefficients, have similar meanings for points through the thickness. Provided that R_1 and R_2 denote the normal radii of curvature of the middle surface, then

$$A_1 = a_1\left(1 + \frac{\zeta}{R_1}\right), \quad A_2 = a_2\left(1 + \frac{\zeta}{R_2}\right) \tag{3}$$

Making use of the preceding formulae one can write the differential elements of area and volume respectively as $dA_0 = a_1 a_2 d\xi_1 d\xi_2$ for the middle surface, $dA_\zeta = A_1 A_2 d\xi_1 d\xi_2$ for the surface at ζ and $dV = A_1 A_2 d\xi_1 d\xi_2 d\zeta$.

B. Strain-Displacement Relations

The non-linear strain components, under the hypothesis of small relative deformations, are defined in curvilinear coordinates,[2] as

$$\begin{aligned}
\varepsilon_{11} &= e_{11} + \frac{1}{2}\left[e_{11}^2 + \left(\frac{1}{2}e_{12} + \omega_3\right)^2 + \left(\frac{1}{2}e_{13} - \omega_2\right)^2\right] \\
\varepsilon_{22} &= e_{22} + \frac{1}{2}\left[e_{22}^2 + \left(\frac{1}{2}e_{12} - \omega_3\right)^2 + \left(\frac{1}{2}e_{23} + \omega_1\right)^2\right] \\
\varepsilon_{12} &= e_{12} + e_{11}\left(\frac{1}{2}e_{12} - \omega_3\right) + e_{22}\left(\frac{1}{2}e_{12} + \omega_3\right) + \left(\frac{1}{2}e_{13} - \omega_2\right)\left(\frac{1}{2}e_{23} + \omega_1\right) \\
\varepsilon_{13} &= e_{13} + e_{11}\left(\frac{1}{2}e_{13} + \omega_2\right) + \left(\frac{1}{2}e_{12} + \omega_3\right)\left(\frac{1}{2}e_{23} - \omega_1\right) \\
\varepsilon_{23} &= e_{23} + e_{22}\left(\frac{1}{2}e_{23} - \omega_1\right) + \left(\frac{1}{2}e_{12} - \omega_3\right)\left(\frac{1}{2}e_{13} + \omega_2\right)
\end{aligned} \tag{4}$$

The expressions in Eqs. (4) are a non-linear combination of those elements that fully describe continuum deformations under the hypothesis of small displacements and rotations, i.e. in the classical linear theory of elasticity (in which $\varepsilon_{ij} \approx e_{ij}$). It is shown in several works,[1,2,6] that the linear components of deformations, in orthogonal curvilinear coordinates, are described using

$$e_{11} = \frac{1}{A_1}\left(\frac{\partial u}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial A_1}{\partial \xi_2}v + \frac{A_1}{R_1}w\right) = \frac{1}{A_1}\left(\frac{\partial u}{\partial \xi_1} + \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}v + \frac{a_1}{R_1}w\right)$$

$$e_{22} = \frac{1}{A_2}\left(\frac{\partial v}{\partial \xi_2} + \frac{1}{A_1}\frac{\partial A_2}{\partial \xi_1}u + \frac{A_2}{R_2}w\right) = \frac{1}{A_2}\left(\frac{\partial v}{\partial \xi_2} + \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}u + \frac{a_2}{R_2}w\right)$$

$$e_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{v}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{u}{A_1}\right) = \frac{1}{A_1}\left(\frac{\partial v}{\partial \xi_1} - \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}u\right) + \frac{1}{A_2}\left(\frac{\partial u}{\partial \xi_2} - \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}v\right)$$

$$e_{13} = A_1\frac{\partial}{\partial \zeta}\left(\frac{u}{A_1}\right) + \frac{1}{A_1}\frac{\partial w}{\partial \xi_1} = \frac{\partial u}{\partial \zeta} + \frac{1}{1+\frac{\zeta}{R_1}}\left(\frac{1}{a_1}\frac{\partial w}{\partial \xi_1} - \frac{u}{R_1}\right) \tag{5}$$

$$e_{23} = \frac{1}{A_2}\frac{\partial w}{\partial \xi_2} + A_2\frac{\partial}{\partial \zeta}\left(\frac{v}{A_2}\right) = \frac{\partial v}{\partial \zeta} + \frac{1}{1+\frac{\zeta}{R_2}}\left(\frac{1}{a_2}\frac{\partial w}{\partial \xi_2} - \frac{v}{R_2}\right)$$

$$2\omega_1 = \frac{1}{A_2}\left[\frac{\partial w}{\partial \xi_2} - \frac{\partial}{\partial \zeta}(A_2 v)\right] = -\frac{\partial v}{\partial \zeta} + \frac{1}{1+\frac{\zeta}{R_2}}\left(\frac{1}{a_2}\frac{\partial w}{\partial \xi_2} - \frac{v}{R_2}\right)$$

$$2\omega_2 = \frac{1}{A_1}\left[\frac{\partial}{\partial \zeta}(A_1 u) - \frac{\partial w}{\partial \xi_1}\right] = \frac{\partial u}{\partial \zeta} - \frac{1}{1+\frac{\zeta}{R_1}}\left(\frac{1}{a_1}\frac{\partial w}{\partial \xi_1} - \frac{u}{R_1}\right)$$

$$2\omega_3 = \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \xi_1}(A_2 v) - \frac{\partial}{\partial \xi_2}(A_1 u)\right] = \frac{1}{A_1}\left(\frac{\partial v}{\partial \xi_1} - \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}u\right) - \frac{1}{A_2}\left(\frac{\partial u}{\partial \xi_2} - \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}v\right)$$

According to the hypothesis described at the beginning of section II, the surface displacements u, v and w are assumed to be

$$u(\xi_1,\xi_2,\zeta,t) = u_0(\xi_1,\xi_2,t) + \zeta\phi_1(\xi_1,\xi_2,t)$$

$$v(\xi_1,\xi_2,\zeta,t) = v_0(\xi_1,\xi_2,t) + \zeta\phi_2(\xi_1,\xi_2,t) \tag{6}$$

$$w(\xi_1,\xi_2,\zeta,t) = w_0(\xi_1,\xi_2,t)$$

Substituting Eqs. (6) into Eqs. (5) enables the linear strain components to be separated into terms depending on displacements and rotations of the middle surface,

$$e_1^0 = \frac{1}{a_1}\left(\frac{\partial u_0}{\partial \xi_1} + \frac{v_0}{a_2}\frac{\partial a_1}{\partial \xi_2} + \frac{a_1}{R_1}w_0\right) \quad e_2^0 = \frac{1}{a_2}\left(\frac{\partial v_0}{\partial \xi_2} + \frac{u_0}{a_1}\frac{\partial a_2}{\partial \xi_1} + \frac{a_2}{R_2}w_0\right)$$

$$\omega_1^0 = \frac{1}{a_1}\left(\frac{\partial v_0}{\partial \xi_1} - \frac{u_0}{a_2}\frac{\partial a_1}{\partial \xi_2}\right) \quad \omega_2^0 = \frac{1}{a_2}\left(\frac{\partial u_0}{\partial \xi_2} - \frac{v_0}{a_1}\frac{\partial a_2}{\partial \xi_1}\right)$$

$$e_4^0 = \frac{1}{a_2}\left(\frac{\partial w_0}{\partial \xi_2} + a_2\phi_2 - \frac{a_2}{R_2}v_0\right) \quad e_5^0 = \frac{1}{a_1}\left(\frac{\partial w_0}{\partial \xi_1} + a_1\phi_1 - \frac{a_1}{R_1}u_0\right)$$

$$\kappa_1^0 = \frac{1}{a_2}\left(\frac{\partial w_0}{\partial \xi_2} - \frac{a_2}{R_2}v_0 - a_2\phi_2\right) \quad \kappa_2^0 = \frac{1}{a_1}\left(-\frac{\partial w_0}{\partial \xi_1} + \frac{a_1}{R_1}u_0 + a_1\phi_1\right)$$

$$e_1^1 = \frac{1}{a_1}\left(\frac{\partial \phi_1}{\partial \xi_1} + \frac{\phi_2}{a_2}\frac{\partial a_1}{\partial \xi_2}\right) \quad e_2^1 = \frac{1}{a_2}\left(\frac{\partial \phi_2}{\partial \xi_2} + \frac{\phi_1}{a_1}\frac{\partial a_2}{\partial \xi_1}\right) \tag{7}$$

$$\omega_1^1 = \frac{1}{a_1}\left(\frac{\partial \phi_2}{\partial \xi_1} - \frac{\phi_1}{a_2}\frac{\partial a_1}{\partial \xi_2}\right) \quad \omega_2^1 = \frac{1}{a_2}\left(\frac{\partial \phi_1}{\partial \xi_2} - \frac{\phi_2}{a_1}\frac{\partial a_2}{\partial \xi_1}\right)$$

$$\kappa_1^1 = -2\frac{\phi_2}{R_2} \quad \kappa_2^1 = +2\frac{\phi_1}{R_1}$$

Herein, superscripts *0* and *1* refer to in-surface and out-of-surface components of linear deformations, respectively. Substituting Eqs. (7) into Eqs. (4) the following expressions for nonlinear strain components are obtained,

$$\varepsilon_{11} = \frac{e_1^0 + \zeta e_1^1}{1+\zeta/R_1} + \frac{1}{2(1+\zeta/R_1)^2}\left[\left(e_1^0 + \zeta e_1^1\right)^2 + \left(\omega_1^0 + \zeta\omega_1^1\right)^2 + \frac{1}{4}\left(e_5^0 - \kappa_2^0 - \zeta\kappa_2^1\right)^2\right]$$

$$\varepsilon_{22} = \frac{e_2^0 + \zeta e_2^1}{1+\zeta/R_2} + \frac{1}{2(1+\zeta/R_2)^2}\left[\left(e_2^0 + \zeta e_2^1\right)^2 + \left(\omega_2^0 + \zeta\omega_2^1\right)^2 + \frac{1}{4}\left(e_4^0 + \kappa_1^0 + \zeta\kappa_1^1\right)^2\right] \tag{8}$$

$$\begin{aligned}\varepsilon_{12} = {} & \frac{\omega_1^0 + \zeta\omega_1^1}{1+\zeta/R_1} + \frac{\omega_2^0 + \zeta\omega_2^1}{1+\zeta/R_2} + \\ & + \frac{1}{(1+\zeta/R_1)(1+\zeta/R_2)}\left[\left(e_1^0 + \zeta e_1^1\right)\left(\omega_2^0 + \zeta\omega_2^1\right) + \left(e_2^0 + \zeta e_2^1\right)\left(\omega_1^0 + \zeta\omega_1^1\right) + \frac{1}{4}\left(e_5^0 - \kappa_2^0 - \zeta\kappa_2^1\right)\left(e_4^0 + \kappa_1^0 + \zeta\kappa_1^1\right)\right]\end{aligned}$$

$$\varepsilon_{13} = \frac{e_5^0}{1+\zeta/R_1} + \frac{1}{2}\left[\frac{\left(e_1^0 + \zeta e_1^1\right)\left(e_5^0 + \kappa_2^0 + \zeta\kappa_2^1\right)}{(1+\zeta/R_1)^2} + \frac{\left(\omega_1^0 + \zeta\omega_1^1\right)\left(e_4^0 - \kappa_1^0 - \zeta\kappa_1^1\right)}{(1+\zeta/R_1)(1+\zeta/R_2)}\right]$$

$$\varepsilon_{23} = \frac{e_4^0}{1+\zeta/R_2} + \frac{1}{2}\left[\frac{\left(e_2^0 + \zeta e_2^1\right)\left(e_4^0 - \kappa_1^0 - \zeta\kappa_1^1\right)}{(1+\zeta/R_2)^2} + \frac{\left(\omega_2^0 + \zeta\omega_2^1\right)\left(e_5^0 + \kappa_2^0 + \zeta\kappa_2^1\right)}{(1+\zeta/R_1)(1+\zeta/R_2)}\right]$$

C. Stress Resultants

The stress resultants acting on the shell element are obtained by integrating each stress component over the thickness[1]; they are

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ Q_{11} \\ M_{11} \\ M_{12} \end{Bmatrix} = \int_{-h/2}^{h/2} \left(1+\zeta/R_2\right) \begin{Bmatrix} \sigma_1 \\ \sigma_6 \\ \sigma_5 K_s \\ \zeta\sigma_1 \\ \zeta\sigma_6 \end{Bmatrix} d\zeta \qquad \begin{Bmatrix} N_{22} \\ N_{21} \\ Q_{22} \\ M_{22} \\ M_{21} \end{Bmatrix} = \int_{-h/2}^{h/2} \left(1+\zeta/R_1\right) \begin{Bmatrix} \sigma_2 \\ \sigma_6 \\ \sigma_4 K_s \\ \zeta\sigma_2 \\ \zeta\sigma_6 \end{Bmatrix} d\zeta \tag{9}$$

where K_s is the shear correction factor used to adjust the discrepancy between the true variation of the transverse shear and that which has been imposed.

D. Constitutive Relations

Suppose that the shell structure is composed of N layers. For each layer the constitutive law is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & \overline{Q}_{26} \\ 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & 0 & 0 & \overline{Q}_{66} \end{bmatrix}^{(k)} \left(\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}^{L} + \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}^{NL} \right) \tag{10}$$

For convenience, in Eq. (10) the strain components vector is split in two parts in which the superscripts L and NL have respectively the meaning of linear and nonlinear. Similarly, the stress resultants are presented as a sum of two vectors corresponding respectively to distributed forces and moments resulting from linear and nonlinear strains. So that, for example, N_{11} will be the sum of N_{11}^{L} and N_{11}^{NL}.

By means of Eqs. (4) one can write

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}^{L} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{23} \\ e_{13} \\ e_{12} \end{Bmatrix} \tag{11}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}^{NL} = \begin{Bmatrix} \frac{1}{2}\left[e_{11}^2 + \left(\frac{1}{2}e_{12}+\omega_3\right)^2 + \left(\frac{1}{2}e_{13}-\omega_2\right)^2 \right] \\ \frac{1}{2}\left[e_{22}^2 + \left(\frac{1}{2}e_{12}-\omega_3\right)^2 + \left(\frac{1}{2}e_{23}+\omega_1\right)^2 \right] \\ e_{22}\left(\frac{1}{2}e_{23}-\omega_1\right) + \left(\frac{1}{2}e_{12}-\omega_3\right)\left(\frac{1}{2}e_{13}+\omega_2\right) \\ e_{11}\left(\frac{1}{2}e_{13}+\omega_2\right) + \left(\frac{1}{2}e_{12}+\omega_3\right)\left(\frac{1}{2}e_{23}-\omega_1\right) \\ e_{11}\left(\frac{1}{2}e_{12}-\omega_3\right) + e_{22}\left(\frac{1}{2}e_{12}+\omega_3\right) + \left(\frac{1}{2}e_{13}-\omega_2\right)\left(\frac{1}{2}e_{23}+\omega_1\right) \end{Bmatrix} \tag{12}$$

Then substituting Eq. (10) back into Eqs. (9) and integrating the subsequent expressions, it is possible to obtain the laminate constitutive relations reported in Eqs. (13)-(15) and Eqs. (22)-(24).

E. Laminate Stiffness Matrix Corresponding to Linear Strains

The constitutive equations are

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ N_{22} \\ N_{21} \end{Bmatrix}^{L} = \begin{bmatrix} A_{11} & A'_{16} & A_{12} & A_{16} \\ A'_{16} & A'_{66} & A_{26} & A_{66} \\ A_{12} & A_{26} & A_{22} & A''_{26} \\ A_{16} & A_{66} & A''_{26} & A''_{66} \end{bmatrix} \begin{Bmatrix} e_1^0 \\ \omega_1^0 \\ e_2^0 \\ \omega_2^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B'_{16} & B_{12} & B_{16} \\ B'_{16} & B'_{66} & B_{26} & B_{66} \\ B_{12} & B_{26} & B_{22} & B''_{26} \\ B_{16} & B_{66} & B''_{26} & B''_{66} \end{bmatrix} \begin{Bmatrix} e_1^1 \\ \omega_1^1 \\ e_2^1 \\ \omega_2^1 \end{Bmatrix} \tag{13}$$

$$\begin{Bmatrix} M_{11} \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix}^{L} = \begin{bmatrix} B_{11} & B'_{16} & B_{12} & B_{16} \\ B'_{16} & B'_{66} & B_{26} & B_{66} \\ B_{12} & B_{26} & B_{22} & B''_{26} \\ B_{16} & B_{66} & B''_{26} & B''_{66} \end{bmatrix} \begin{Bmatrix} e_1^0 \\ \omega_1^0 \\ e_2^0 \\ \omega_2^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D'_{16} & D_{12} & D_{16} \\ D'_{16} & D'_{66} & D_{26} & D_{66} \\ D_{12} & D_{26} & D_{22} & D''_{26} \\ D_{16} & D_{66} & D''_{26} & D''_{66} \end{bmatrix} \begin{Bmatrix} e_1^1 \\ \omega_1^1 \\ e_2^1 \\ \omega_2^1 \end{Bmatrix} \tag{14}$$

$$\begin{Bmatrix} Q_{22} \\ Q_{11} \end{Bmatrix}^{L} = K_s \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} e_4^0 \\ e_5^0 \end{Bmatrix} \tag{15}$$

or in a more compact form,

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ Q_{22} \\ N_{22} \\ N_{21} \\ Q_{11} \\ M_{11} \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix}^{L} = \begin{bmatrix} A_{11} & A'_{16} & 0 & A_{12} & A_{16} & 0 & B_{11} & B'_{16} & B_{12} & B_{16} \\ A'_{16} & A'_{66} & 0 & A_{26} & A_{66} & 0 & B'_{16} & B'_{66} & B_{26} & B_{66} \\ 0 & 0 & K_s A_{44} & 0 & 0 & K_s A_{45} & 0 & 0 & 0 & 0 \\ A_{12} & A_{26} & 0 & A_{22} & A''_{26} & 0 & B_{12} & B_{26} & B_{22} & B''_{26} \\ A_{16} & A_{66} & 0 & A''_{26} & A''_{66} & 0 & B_{16} & B_{66} & B''_{26} & B''_{66} \\ 0 & 0 & K_s A_{45} & 0 & 0 & K_s A_{55} & 0 & 0 & 0 & 0 \\ B_{11} & B'_{16} & 0 & B_{12} & B_{16} & 0 & D_{11} & D'_{16} & D_{12} & D_{16} \\ B'_{16} & B'_{66} & 0 & B_{26} & B_{66} & 0 & D'_{16} & D'_{66} & D_{26} & D_{66} \\ B_{12} & B_{26} & 0 & B_{22} & B''_{26} & 0 & D_{12} & D_{26} & D_{22} & D''_{26} \\ B_{16} & B_{66} & 0 & B''_{26} & B''_{66} & 0 & D_{16} & D_{66} & D''_{26} & D''_{66} \end{bmatrix} \begin{Bmatrix} e_1^0 \\ \omega_1^0 \\ e_4^0 \\ e_2^0 \\ \omega_2^0 \\ e_5^0 \\ e_1^1 \\ \omega_1^1 \\ e_2^1 \\ \omega_2^1 \end{Bmatrix} \tag{16}$$

The elements of Eqs. (13)-(16), which are due to the linear part of the strain components, can be calculated using the following relations,

$$\underline{\underline{A}} = \begin{bmatrix} A_{11} & A'_{16} & A_{12} & A_{16} \\ A'_{16} & A'_{66} & A_{26} & A_{66} \\ A_{12} & A_{26} & A_{22} & A''_{26} \\ A_{16} & A_{66} & A''_{26} & A''_{66} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} a_{R_1}^L \overline{Q}_{11} & a_{R_1}^L \overline{Q}_{16} & c^L \overline{Q}_{12} & c^L \overline{Q}_{16} \\ a_{R_1}^L \overline{Q}_{16} & a_{R_1}^L \overline{Q}_{66} & c^L \overline{Q}_{26} & c^L \overline{Q}_{66} \\ c^L \overline{Q}_{12} & c^L \overline{Q}_{16} & a_{R_2}^L \overline{Q}_{22} & a_{R_2}^L \overline{Q}_{26} \\ c^L \overline{Q}_{26} & c^L \overline{Q}_{66} & a_{R_2}^L \overline{Q}_{26} & a_{R_2}^L \overline{Q}_{66} \end{bmatrix}^{(k)} \tag{17}$$

$$\underline{\underline{B}} = \begin{bmatrix} B_{11} & B'_{16} & B_{12} & B_{16} \\ B'_{16} & B'_{66} & B_{26} & B_{66} \\ B_{12} & B_{26} & B_{22} & B''_{26} \\ B_{16} & B_{66} & B''_{26} & B''_{66} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} b_{R_1}^L \overline{Q}_{11} & b_{R_1}^L \overline{Q}_{16} & d^L \overline{Q}_{12} & d^L \overline{Q}_{16} \\ b_{R_1}^L \overline{Q}_{16} & b_{R_1}^L \overline{Q}_{66} & d^L \overline{Q}_{26} & d^L \overline{Q}_{66} \\ d^L \overline{Q}_{12} & d^L \overline{Q}_{16} & b_{R_2}^L \overline{Q}_{22} & b_{R_2}^L \overline{Q}_{26} \\ d^L \overline{Q}_{26} & d^L \overline{Q}_{66} & b_{R_2}^L \overline{Q}_{26} & b_{R_2}^L \overline{Q}_{66} \end{bmatrix}^{(k)} \tag{18}$$

$$\underline{\underline{D}} = \begin{bmatrix} D_{11} & D'_{16} & D_{12} & D_{16} \\ D'_{16} & D'_{66} & D_{26} & D_{66} \\ D_{12} & D_{26} & D_{22} & D''_{26} \\ D_{16} & D_{66} & D''_{26} & D''_{66} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} e^L_{R_1}\overline{Q}_{11} & e^L_{R_1}\overline{Q}_{16} & f^L\overline{Q}_{12} & f^L\overline{Q}_{16} \\ e^L_{R_1}\overline{Q}_{16} & e^L_{R_1}\overline{Q}_{66} & f^L\overline{Q}_{26} & f^L\overline{Q}_{66} \\ f^L\overline{Q}_{12} & f^L\overline{Q}_{16} & e^L_{R_2}\overline{Q}_{22} & e^L_{R_2}\overline{Q}_{26} \\ f^L\overline{Q}_{26} & f^L\overline{Q}_{66} & e^L_{R_2}\overline{Q}_{26} & e^L_{R_2}\overline{Q}_{66} \end{bmatrix}^{(k)} \tag{19}$$

$$\begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} a^L_{R_2}\overline{Q}_{44} & c^L\overline{Q}_{45} \\ c^L\overline{Q}_{45} & a^L_{R_1}\overline{Q}_{55} \end{bmatrix}^{(k)} \tag{20}$$

and

$$a^L_{R_1} = \frac{R_1}{R_2}\left[(\zeta_{k+1} - \zeta_k) + (R_2 - R_1)\ln\left(\frac{R_1 + \zeta_{k+1}}{R_1 + \zeta_k}\right)\right]$$

$$b^L_{R_1} = \frac{R_1}{R_2}\left[\frac{1}{2}(\zeta^2_{k+1} - \zeta^2_k) + (R_2 - R_1)(\zeta_{k+1} - \zeta_k) - R_1(R_2 - R_1)\ln\left(\frac{R_1 + \zeta_{k+1}}{R_1 + \zeta_k}\right)\right]$$

$$c^L = (\zeta_{k+1} - \zeta_k) \tag{21}$$

$$d^L = \frac{1}{2}(\zeta^2_{k+1} - \zeta^2_k)$$

$$e^L_{R_1} = \frac{R_1}{R_2}\left[\frac{1}{3}(\zeta^3_{k+1} - \zeta^3_k) + \frac{1}{2}(R_2 - R_1)(\zeta^2_{k+1} - \zeta^2_k) - R_1(R_2 - R_1)(\zeta_{k+1} - \zeta_k) + R_1^2(R_2 - R_1)\ln\left(\frac{R_1 + \zeta_{k+1}}{R_1 + \zeta_k}\right)\right]$$

$$f^L = \frac{1}{3}(\zeta^3_{k+1} - \zeta^3_k)$$

Similar coefficients with R_2 as a subscript can be obtained simply interchanging subscripts *1* and *2*. It is of significance that the stiffness matrices remain symmetric but have been enlarged to a 4x4 matrix system. Note, symmetry has been achieved by decomposing shear strains into two separate components.

F. Laminate Stiffness Matrix Corresponding to Nonlinear Strains

Similarly, it is possible to obtain the part of the constitutive equations due to the nonlinear strain components,

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ N_{22} \\ N_{21} \end{Bmatrix}^{NL} = \frac{1}{4}\begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \\ \mathrm{A}_{41} & \mathrm{A}_{42} & \mathrm{A}_{43} \end{bmatrix} \begin{Bmatrix} \frac{1}{2}\left(4e_1^{0^2} + 4\omega_1^{0^2} + e_5^{0^2} + \kappa_2^{0^2} - 2e_5^0\kappa_2^0\right) \\ \frac{1}{2}\left(4e_2^{0^2} + 4\omega_2^{0^2} + e_4^{0^2} + \kappa_1^{0^2} + 2e_4^0\kappa_1^0\right) \\ 4\omega_2^0 e_1^0 + 4\omega_1^0 e_2^0 + e_5^0 e_4^0 + e_5^0\kappa_1^0 - e_4^0\kappa_2^0 - \kappa_1^0\kappa_2^0 \end{Bmatrix} + \tag{22}$$

$$+\frac{1}{4}\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} \end{bmatrix} \begin{Bmatrix} 4e_1^0 e_1^1 + 4\omega_1^0\omega_1^1 - e_5^0\kappa_2^1 + \kappa_2^0\kappa_2^1 \\ 4e_2^0 e_2^1 + 4\omega_2^0\omega_2^1 + e_4^0\kappa_1^1 + \kappa_1^0\kappa_1^1 \\ 4\left(\omega_2^1 e_1^0 + \omega_2^0 e_1^1 + \omega_1^1 e_2^0 + \omega_1^0 e_2^1\right) + \kappa_1^1\left(e_5^0 - \kappa_2^0\right) - \kappa_2^1\left(e_4^0 + \kappa_1^0\right) \end{Bmatrix} +$$

$$
+\frac{1}{4}\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \end{bmatrix}\begin{Bmatrix} \frac{1}{2}\left(4e_1^{1^2}+4\omega_1^{1^2}+\kappa_2^{1^2}\right) \\ \frac{1}{2}\left(4e_2^{1^2}+4\omega_2^{1^2}+\kappa_1^{1^2}\right) \\ 4\omega_2^1 e_1^1+4\omega_1^1 e_2^1-\kappa_1^1\kappa_2^1 \end{Bmatrix}
$$

$$
\begin{Bmatrix} M_{11} \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix}^{NL}=\frac{1}{4}\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} \end{bmatrix}\begin{Bmatrix} \frac{1}{2}\left(4e_1^{0^2}+4\omega_1^{0^2}+e_5^{0^2}+\kappa_2^{0^2}-2e_5^0\kappa_2^0\right) \\ \frac{1}{2}\left(4e_2^{0^2}+4\omega_2^{0^2}+e_4^{0^2}+\kappa_1^{0^2}+2e_4^0\kappa_1^0\right) \\ 4\omega_2^0 e_1^0+4\omega_1^0 e_2^0+e_5^0 e_4^0+e_5^0\kappa_1^0-e_4^0\kappa_2^0-\kappa_1^0\kappa_2^0 \end{Bmatrix}+
$$

$$
+\frac{1}{4}\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \end{bmatrix}\begin{Bmatrix} 4e_1^0 e_1^1+4\omega_1^0\omega_1^1-e_5^0\kappa_2^1+\kappa_2^0\kappa_2^1 \\ 4e_2^0 e_2^1+4\omega_2^0\omega_2^1+e_4^0\kappa_1^1+\kappa_1^0\kappa_1^1 \\ 4\left(\omega_2^1 e_1^0+\omega_2^0 e_1^1+\omega_1^1 e_2^0+\omega_1^0 e_2^1\right)+\kappa_1^1\left(e_5^0-\kappa_2^0\right)-\kappa_2^1\left(e_4^0+\kappa_1^0\right) \end{Bmatrix}+ \tag{23}
$$

$$
+\frac{1}{4}\begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} \end{bmatrix}\begin{Bmatrix} \frac{1}{2}\left(4e_1^{1^2}+4\omega_1^{1^2}+\kappa_2^{1^2}\right) \\ \frac{1}{2}\left(4e_2^{1^2}+4\omega_2^{1^2}+\kappa_1^{1^2}\right) \\ 4\omega_2^1 e_1^1+4\omega_1^1 e_2^1-\kappa_1^1\kappa_2^1 \end{Bmatrix}
$$

and

$$
\begin{Bmatrix} Q_{22} \\ Q_{11} \end{Bmatrix}^{NL}=\frac{K_s}{2}\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \end{bmatrix}\begin{Bmatrix} e_2^0 e_4^0-e_2^0\kappa_1^0 \\ \omega_2^0 e_5^0+\omega_2^0\kappa_2^0 \\ e_1^0 e_5^0+e_1^0\kappa_2^0 \\ \omega_1^0 e_4^0-\omega_1^0\kappa_1^0 \end{Bmatrix}+
$$

$$
+\frac{K_s}{2}\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \end{bmatrix}\begin{Bmatrix} e_2^1 e_4^0-e_2^0\kappa_1^1-e_2^1\kappa_1^0 \\ \omega_2^0\kappa_2^1+\omega_2^1 e_5^0+\omega_2^1\kappa_2^0 \\ e_1^1 e_5^0+e_1^0\kappa_2^1+e_1^1\kappa_2^0 \\ \omega_1^1 e_4^0-\omega_1^0\kappa_1^1-\omega_1^1\kappa_1^0 \end{Bmatrix}+ \tag{24}
$$

$$
+\frac{K_s}{2}\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \end{bmatrix}\begin{Bmatrix} -e_2^1\kappa_1^1 \\ \omega_2^1\kappa_2^1 \\ e_1^1\kappa_2^1 \\ -\omega_1^1\kappa_1^1 \end{Bmatrix}
$$

The coefficients of Eqs. (22)-(24), which are due to the nonlinear part of the strain components, can be calculated using the following relations

$$
\underline{\underline{A}}=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix}=\sum_{k=1}^{N}\begin{bmatrix} \overline{Q}_{11}a_{R_1}^{NL} & \overline{Q}_{12}d_{R_2}^{NL} & \overline{Q}_{16}d_{R_1}^{NL} \\ \overline{Q}_{16}a_{R_1}^{NL} & \overline{Q}_{26}d_{R_2}^{NL} & \overline{Q}_{66}d_{R_1}^{NL} \\ \overline{Q}_{12}d_{R_1}^{NL} & \overline{Q}_{22}a_{R_2}^{NL} & \overline{Q}_{26}d_{R_2}^{NL} \\ \overline{Q}_{16}d_{R_1}^{NL} & \overline{Q}_{26}a_{R_2}^{NL} & \overline{Q}_{66}d_{R_2}^{NL} \end{bmatrix}^{(k)} \tag{25}
$$

$$\underline{\underline{B}} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} \overline{Q}_{11} c_{R_1}^{NL} & \overline{Q}_{12} f_{R_2}^{NL} & \overline{Q}_{16} f_{R_1}^{NL} \\ \overline{Q}_{16} c_{R_1}^{NL} & \overline{Q}_{26} f_{R_2}^{NL} & \overline{Q}_{66} f_{R_1}^{NL} \\ \overline{Q}_{12} f_{R_1}^{NL} & \overline{Q}_{22} c_{R_2}^{NL} & \overline{Q}_{26} f_{R_2}^{NL} \\ \overline{Q}_{16} f_{R_1}^{NL} & \overline{Q}_{26} c_{R_2}^{NL} & \overline{Q}_{66} f_{R_2}^{NL} \end{bmatrix}^{(k)} \tag{26}$$

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} \overline{Q}_{11} b_{R_1}^{NL} & \overline{Q}_{12} e_{R_2}^{NL} & \overline{Q}_{16} e_{R_1}^{NL} \\ \overline{Q}_{16} b_{R_1}^{NL} & \overline{Q}_{26} e_{R_2}^{NL} & \overline{Q}_{66} e_{R_1}^{NL} \\ \overline{Q}_{12} e_{R_1}^{NL} & \overline{Q}_{22} b_{R_2}^{NL} & \overline{Q}_{26} e_{R_2}^{NL} \\ \overline{Q}_{16} e_{R_1}^{NL} & \overline{Q}_{26} b_{R_2}^{NL} & \overline{Q}_{66} e_{R_2}^{NL} \end{bmatrix}^{(k)} \tag{27}$$

$$\underline{\underline{\Delta}} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} \overline{Q}_{11} g_{R_1}^{NL} & \overline{Q}_{12} h_{R_2}^{NL} & \overline{Q}_{16} h_{R_1}^{NL} \\ \overline{Q}_{16} g_{R_1}^{NL} & \overline{Q}_{26} h_{R_2}^{NL} & \overline{Q}_{66} h_{R_1}^{NL} \\ \overline{Q}_{12} h_{R_1}^{NL} & \overline{Q}_{22} g_{R_2}^{NL} & \overline{Q}_{26} h_{R_2}^{NL} \\ \overline{Q}_{16} h_{R_1}^{NL} & \overline{Q}_{26} g_{R_2}^{NL} & \overline{Q}_{66} h_{R_2}^{NL} \end{bmatrix}^{(k)} \tag{28}$$

$$\underline{\underline{E}} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} \overline{Q}_{44} a_{R_2}^{NL} & \overline{Q}_{44} d_{R_2}^{NL} & \overline{Q}_{45} d_{R_1}^{NL} & \overline{Q}_{45} d_{R_2}^{NL} \\ \overline{Q}_{45} d_{R_2}^{NL} & \overline{Q}_{45} d_{R_1}^{NL} & \overline{Q}_{55} a_{R_1}^{NL} & \overline{Q}_{55} d_{R_1}^{NL} \end{bmatrix}^{(k)} \tag{29}$$

$$\underline{\underline{Z}} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} \overline{Q}_{44} b_{R_2}^{NL} & \overline{Q}_{44} e_{R_2}^{NL} & \overline{Q}_{45} e_{R_1}^{NL} & \overline{Q}_{45} e_{R_2}^{NL} \\ \overline{Q}_{45} e_{R_2}^{NL} & \overline{Q}_{45} e_{R_1}^{NL} & \overline{Q}_{55} b_{R_1}^{NL} & \overline{Q}_{55} e_{R_1}^{NL} \end{bmatrix}^{(k)} \tag{30}$$

$$\underline{\underline{H}} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} \overline{Q}_{44} c_{R_2}^{NL} & \overline{Q}_{44} f_{R_2}^{NL} & \overline{Q}_{45} f_{R_1}^{NL} & \overline{Q}_{45} f_{R_2}^{NL} \\ \overline{Q}_{45} f_{R_2}^{NL} & \overline{Q}_{45} f_{R_1}^{NL} & \overline{Q}_{55} c_{R_1}^{NL} & \overline{Q}_{55} f_{R_1}^{NL} \end{bmatrix}^{(k)} \tag{31}$$

where,

$$a_{R_1}^{NL} = \frac{R_1^2}{R_2}\left[\ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right) + (R_2-R_1)\frac{(\zeta_{k+1}-\zeta_k)}{(R_1+\zeta_{k+1})(R_1+\zeta_k)}\right]$$

$$b_{R_1}^{NL} = \frac{R_1^2}{R_2}\left[(\zeta_{k+1}-\zeta_k) + (R_2-2R_1)\ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right) - R_1(R_2-R_1)\frac{(\zeta_{k+1}-\zeta_k)}{(R_1+\zeta_{k+1})(R_1+\zeta_k)}\right]$$

$$\begin{aligned} c_{R_1}^{NL} = \frac{R_1^2}{R_2}\Bigg[& \frac{1}{2}(\zeta_{k+1}^2-\zeta_k^2) + (R_2-2R_1)(\zeta_{k+1}-\zeta_k) + \\ & + R_1(3R_1-2R_2)\ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right) + R_1^2(R_2-R_1)\frac{(\zeta_{k+1}-\zeta_k)}{(R_1+\zeta_{k+1})(R_1+\zeta_k)}\Bigg] \end{aligned}$$

$$d_{R_1}^{NL} = R_1 \ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right) \tag{32}$$

$$e_{R_1}^{NL} = R_1\left[\left(\zeta_{k+1}-\zeta_k\right)-R_1\ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right)\right]$$

$$f_{R_1}^{NL} = R_1\left[\frac{1}{2}\left(\zeta_{k+1}^2-\zeta_k^2\right)-R_1\left(\zeta_{k+1}-\zeta_k\right)+R_1^2\ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right)\right]$$

$$g_{R_1}^{NL} = \frac{R_1^2}{R_2}\left[\frac{1}{3}\left(\zeta_{k+1}^3-\zeta_k^3\right)+\left(\frac{R_2}{2}-R_1\right)\left(\zeta_{k+1}^2-\zeta_k^2\right)+R_1\left(3R_1-2R_2\right)\left(\zeta_{k+1}-\zeta_k\right)+\right.$$
$$\left.+R_1^2\left(3R_2-4R_1\right)\ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right)-R_1^3\left(R_2-R_1\right)\frac{\left(\zeta_{k+1}-\zeta_k\right)}{\left(R_1+\zeta_{k+1}\right)\left(R_1+\zeta_k\right)}\right]$$

$$h_{R_1}^{NL} = R_1\left[\frac{1}{3}\left(\zeta_{k+1}^3-\zeta_k^3\right)-\frac{1}{2}R_1\left(\zeta_{k+1}^2-\zeta_k^2\right)+R_1^2\left(\zeta_{k+1}-\zeta_k\right)-R_1^3\ln\left(\frac{R_1+\zeta_{k+1}}{R_1+\zeta_k}\right)\right]$$

III. Equations of Motion

The following six equations of equilibrium are well-known and widely accepted.[1,6] They reflect the equilibrium of the middle surface when a transverse load q is applied,

$$\frac{\partial}{\partial\xi_1}\left(a_2N_{11}\right)+\frac{\partial}{\partial\xi_2}\left(a_1N_{21}\right)-N_{22}\frac{\partial a_2}{\partial\xi_1}+N_{12}\frac{\partial a_1}{\partial\xi_2}+\frac{a_1a_2}{R_1}Q_1 = a_1a_2\left(I_0\frac{\partial^2u_0}{\partial t^2}+I_1\frac{\partial^2\phi_1}{\partial t^2}\right)$$

$$\frac{\partial}{\partial\xi_1}\left(a_2N_{12}\right)+\frac{\partial}{\partial\xi_2}\left(a_1N_{22}\right)-N_{11}\frac{\partial a_1}{\partial\xi_2}+N_{21}\frac{\partial a_2}{\partial\xi_1}+\frac{a_1a_2}{R_2}Q_2 = a_1a_2\left(I_0\frac{\partial^2v_0}{\partial t^2}+I_1\frac{\partial^2\phi_2}{\partial t^2}\right)$$

$$\frac{\partial}{\partial\xi_1}\left(a_2Q_1\right)+\frac{\partial}{\partial\xi_2}\left(a_1Q_2\right)-a_1a_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right) = qa_1a_2+a_1a_2I_0\frac{\partial^2w_0}{\partial t^2} \tag{33}$$

$$\frac{\partial}{\partial\xi_1}\left(a_2M_{11}\right)+\frac{\partial}{\partial\xi_2}\left(a_1M_{21}\right)-M_{22}\frac{\partial a_2}{\partial\xi_1}+M_{12}\frac{\partial a_1}{\partial\xi_2}-a_1a_2Q_1 = a_1a_2\left(I_1\frac{\partial^2u_0}{\partial t^2}+I_2\frac{\partial^2\phi_1}{\partial t^2}\right)$$

$$\frac{\partial}{\partial\xi_1}\left(a_2M_{12}\right)+\frac{\partial}{\partial\xi_2}\left(a_1M_{22}\right)-M_{11}\frac{\partial a_1}{\partial\xi_2}+M_{21}\frac{\partial a_2}{\partial\xi_1}-a_1a_2Q_2 = a_1a_2\left(I_1\frac{\partial^2v_0}{\partial t^2}+I_2\frac{\partial^2\phi_2}{\partial t^2}\right)$$

$$\frac{M_{12}}{R_2}-\frac{M_{21}}{R_1}+N_{21}-N_{12} = 0\,.$$

The mass inertias I_0, I_1, I_2 are calculated using

$$I_i = \sum_{k=1}^{N}\int_{-h/2}^{h/2}\rho^{(k)}\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)\zeta^i d\zeta \qquad \left(i = 0, 1, 2\right) \tag{34}$$

where ρ is the mass density.

Due to the definition of the stress resultants, the last expression in Eq. (33), concerning "drilling" equilibrium, is always satisfied and, for this reason, it is usually not considered in deriving the differential equations relating displacements and applied loads. It is noted that drilling equilibrium is always satisfied using Eq. (9). However, in deriving the stress resultants, approximate expressions for the displacements field have been used. It has been demonstrated that this discrepancy would lead to nonzero stress resultants corresponding to a small rigid body rotation and that Eq. (16) is thus in need of some modification.[16,17] Analytically, this is done by modifying the strain-displacements relations in Eq. (16) by replacing those terms without tilde using the following terms with tilde,

$$\tilde{\omega}_1^0 = \frac{1}{a_1}\left(\frac{\partial v_0}{\partial \xi_1} - \frac{u_0}{a_2}\frac{\partial a_1}{\partial \xi_2}\right) - \phi_n \qquad \tilde{\omega}_2^0 = \frac{1}{a_2}\left(\frac{\partial u_0}{\partial \xi_2} - \frac{v_0}{a_1}\frac{\partial a_2}{\partial \xi_1}\right) + \phi_n$$

$$\tilde{\omega}_1^1 = \frac{1}{a_1}\left(\frac{\partial \phi_2}{\partial \xi_1} - \frac{\phi_1}{a_2}\frac{\partial a_1}{\partial \xi_2}\right) - \frac{\phi_n}{R_1} \qquad \tilde{\omega}_2^1 = \frac{1}{a_2}\left(\frac{\partial \phi_1}{\partial \xi_2} - \frac{\phi_2}{a_1}\frac{\partial a_2}{\partial \xi_1}\right) + \frac{\phi_n}{R_2} \tag{35}$$

Here the term ϕ_n is the third component of the curl of the shell middle surface displacements field, hence

$$\phi_n = \frac{1}{2a_1a_2}\left[\frac{\partial}{\partial \xi_1}(a_2 v_0) - \frac{\partial}{\partial \xi_2}(a_1 u_0)\right] \tag{36}$$

IV. Case Studies

One of the novelties introduced in this work is that the term $(1+\zeta/R_i)$ has been retained in the derivation of the shell model. This term is typically neglected because, for the range of applicability of any shell theory based on an approximation of the shell as a two dimensional structure, the quantity ζ/R_i is small if compared to unity.

In the following sections brief examples of application of the developed theory will be provided. The expressions of the stiffnesses are presented as functions of the geometry of the shell in its orthogonal curvilinear system i.e. of the normal radii of curvature. These functions therefore represent a point to point mapping between the structure's idealized domain and stiffness. In other words, they allow one to calculate analytically the stiffnesses of each point within a structure.

It is later shown that, despite the confirmation of the approximation that ζ/R_i is small compared to unity, by neglecting this term entails the loss of crucial pieces of information. Indeed, the geometry of a shell structure can affect the stiffness matrix, by introducing coupling terms even for symmetric laminates. This effect is readily explained by simplifying the expressions in Eq. (21). For instance, consider a generic shell of thickness h and assume that the material is isotropic. A series expansion of Eqs. (21) yields the following relationships:

$$a_{R_1}^L \approx a_{R_2}^L \approx h - \frac{1}{12}\frac{h^3}{R_1R_2} + \frac{1}{12}\frac{h^3}{R_1^2}$$

$$b_{R_1}^L \approx b_{R_2}^L \approx \frac{1}{12}\frac{h^3}{R_2} - \frac{1}{12}\frac{h^3}{R_1} \tag{37}$$

$$e_{R_1}^L \approx \frac{1}{12}h^3 \qquad e_{R_2}^L \approx \frac{1}{12}h^3 - \frac{1}{80}\frac{h^5}{R_1R_2} + \frac{1}{80}\frac{h^5}{R_2^2}$$

Equation (37) gives an idea of the order of magnitude of the difference between classic lamination theory (CLT) stiffnesses and the ones herein presented and also shows that this difference depends on the sign of R_1R_2. By comparing the latter expressions to the classic case, in which:

$$a_{R_1}^L \approx a_{R_2}^L \approx h \qquad b_{R_1}^L \approx b_{R_2}^L \approx 0 \qquad e_{R_1}^L \approx e_{R_2}^L \approx \frac{1}{12}h^3 \tag{38}$$

it becomes clear that although there is a small, even if non-uniform, difference for the A_{ij} and D_{ij} terms, the B_{ij} terms are different from zero, of the order of D_{ij}/R and magnified for structures in which $R_1R_2<0$. For composite structures the difference is then expected to be of the same order of magnitude.

In the following sections, stiffness matrices resulting from Eqs. (17) to (20) are presented in comparison with the equivalent classic matrices, as in Ref. 1. The structures in the examples are all assumed to be made from layers with material properties:

$$E_1 = 206.8\,GPa\,,\quad E_2 = 20.7\,GPa$$

$$G_{12} = G_{13} = 10.3\,GPa\,,\quad G_{23} = 4.1\,GPa$$

$$\nu_{12} = 0.25$$

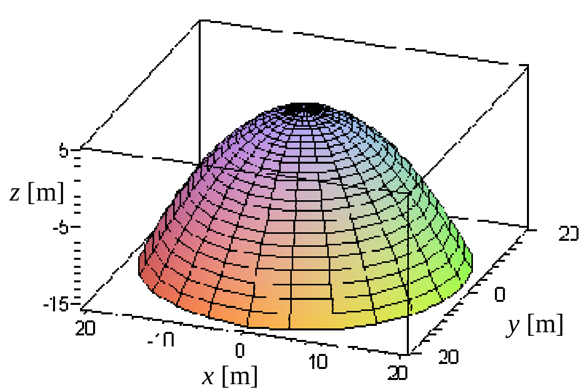
and a symmetric lay-up with stacking sequence [45 30 90 0]$_s$. For a useful comparison it is worth noting that according to Ref. 1 $N_{12}=N_{21}$, $M_{12}=M_{21}$ and:

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} e_1^0 \\ e_2^0 \\ \omega_1^0+\omega_2^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} e_1^1 \\ e_2^1 \\ \omega_1^1+\omega_2^1 \end{Bmatrix} \tag{39}$$

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} e_1^0 \\ e_2^0 \\ \omega_1^0+\omega_2^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} e_1^1 \\ e_2^1 \\ \omega_1^1+\omega_2^1 \end{Bmatrix} \tag{40}$$

A. Paraboloid of Revolution ($R_1R_2>0$, $h=2\mathrm{m}$)

Figure 1 represents a structure whose shape is a paraboloid of revolution. Here δ represents the vertical coordinate along the generators in the orthogonal curvilinear coordinate system lying on the paraboloid.



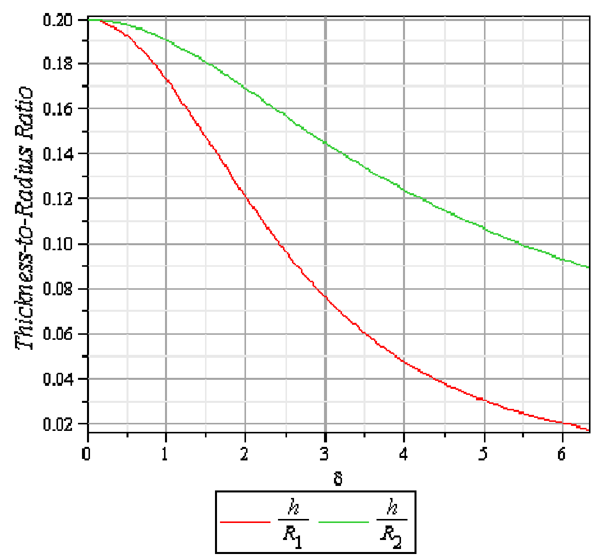


Figure 1. Paraboloid of Revolution geometry.

Because of the symmetry of the structure the elements of $\underline{\underline{B}}$,

$$\underline{\underline{B}}\left(R_1(\delta), R_2(\delta)\right) = \begin{bmatrix} B_{11} & B'_{16} & 0 & 0 \\ B'_{16} & B'_{66} & 0 & 0 \\ 0 & 0 & B_{22} & B''_{26} \\ 0 & 0 & B''_{26} & B''_{66} \end{bmatrix}; \quad \underline{\underline{B}}(CLT) = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{41}$$

are a function of δ only. As expected, the new B_{ij} elements in Eq. (41) are of the order of D_{ij}/R and, as shown by the $b^L_{R_i}$ terms in Eq. (37), are proportional to $\left(h/R_2 - h/R_1\right)$, i.e. to the difference between the green and red curves on the right-hand side of Fig. 1. The distribution of B_{ij}, as a function of δ, is shown in Fig. 2.

Figure 2 shows that there is a small but non-negligible coupling between the paraboloid in-surface and out-of-surface mechanical behavior. This coupling, which arises solely from the geometry of the structure, affects some of the terms in $\underline{\underline{B}}$. Due to the proportionality of these stiffnesses to the difference between the thickness to radius ratios, the coupling is maximum where this difference is at its peak. Note, that its value is close to zero at the top of the structure and tends asymptotically to a significant value for large values of δ (i.e. $\delta > 6$).

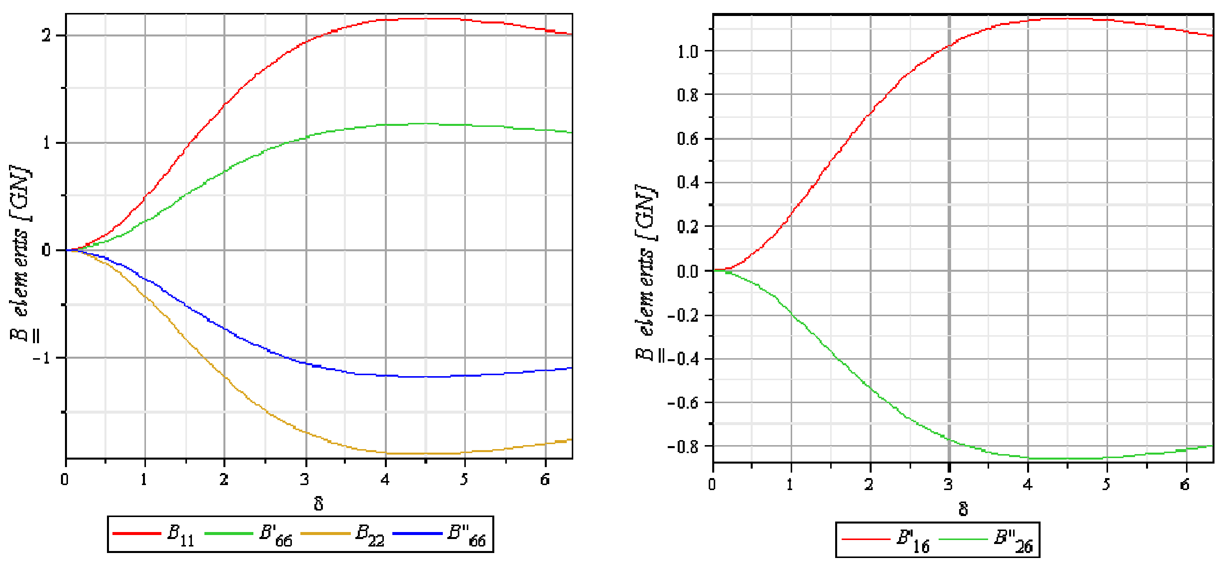


Figure 2. B matrix elements.

B. Spherical Shell ($R_1 = R_2$)

In this case, where $R_1 = R_2$, any geometric effect on the stiffnesses disappears.

C. Cylindrical Shell ($1/R_1 = 0$, $R_2 = 10\text{m}$, $h/R_2 = 0.1$)

For cylindrical structures the effect of curvatures on $\underline{\underline{B}}$ is proportional to h/R_2. The stiffness matrices developed herein are compared with the classical (flat plate) below,

$$\underline{\underline{A}} = \begin{bmatrix} A_{11} & A'_{16} & A_{12} & A_{16} \\ A'_{16} & A'_{66} & A_{26} & A_{66} \\ A_{12} & A_{26} & A_{22} & A''_{26} \\ A_{16} & A_{66} & A''_{26} & A''_{66} \end{bmatrix} = \begin{bmatrix} 106.81 & 26.64 & 24.58 & 26.64 \\ 26.64 & 29.72 & 17.05 & 29.72 \\ 24.58 & 17.05 & 83.45 & 17.08 \\ 26.64 & 29.72 & 17.08 & 29.76 \end{bmatrix}; \quad \underline{\underline{A}}(CLT) = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} = \begin{bmatrix} 106.81 & 24.58 & 26.64 \\ 24.58 & 83.39 & 17.05 \\ 26.64 & 17.05 & 29.72 \end{bmatrix}$$

$$\underline{\underline{B}} = \begin{bmatrix} B_{11} & B'_{16} & B_{12} & B_{16} \\ B'_{16} & B'_{66} & B_{26} & B_{66} \\ B_{12} & B_{26} & B_{22} & B''_{26} \\ B_{16} & B_{66} & B''_{26} & B''_{66} \end{bmatrix} = \begin{bmatrix} 0.70 & 0.37 & 0 & 0 \\ 0.37 & 0.38 & 0 & 0 \\ 0 & 0 & -0.62 & -0.28 \\ 0 & 0 & -0.28 & -0.38 \end{bmatrix}; \quad \underline{\underline{B}}(CLT) = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{42}$$

$$\underline{\underline{D}} = \begin{bmatrix} D_{11} & D'_{16} & D_{12} & D_{16} \\ D'_{16} & D'_{66} & D_{26} & D_{66} \\ D_{12} & D_{26} & D_{22} & D''_{26} \\ D_{16} & D_{66} & D''_{26} & D''_{66} \end{bmatrix} = \begin{bmatrix} 7.01 & 3.73 & 3.39 & 3.73 \\ 3.73 & 3.82 & 2.79 & 3.82 \\ 3.39 & 2.79 & 6.17 & 2.79 \\ 3.73 & 3.82 & 2.79 & 3.82 \end{bmatrix}; \quad \underline{\underline{D}}(CLT) = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} = \begin{bmatrix} 7.01 & 3.39 & 3.73 \\ 3.39 & 6.15 & 2.78 \\ 3.73 & 2.78 & 3.82 \end{bmatrix}$$

$$\begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} = \begin{bmatrix} 6.86 & 1.45 \\ 1.45 & 7.63 \end{bmatrix}; \quad \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix}(CLT) = \begin{bmatrix} 6.85 & 1.45 \\ 1.45 & 7.63 \end{bmatrix}$$

Once again, the effect of curvatures on $\underline{\underline{A}}$ and $\underline{\underline{D}}$ is small but there is a significant effect on $\underline{\underline{B}}$.

D. Conical Shell ($1/R_1 = 0$, $R_2 > 0$, $h = 1\mathrm{m}$)

Figure 3 represents a structure whose shape is a truncated cone. In this example δ coincides with the vertical coordinate along the generators in the orthogonal curvilinear coordinate system lying on the cone. The elements of the coupling stiffness matrix are

$$\underline{\underline{B}}(R_2(\delta)) = \begin{bmatrix} B_{11} & B'_{16} & 0 & 0 \\ B'_{16} & B'_{66} & 0 & 0 \\ 0 & 0 & B_{22} & B''_{26} \\ 0 & 0 & B''_{26} & B''_{66} \end{bmatrix}; \quad \underline{\underline{B}}(CLT) = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{43}$$

As indicated in Eq. (43), $\underline{\underline{B}}$ is partially populated. From the $b^L_{R_i}$ terms in Eq (37), the effect of the curvatures on $\underline{\underline{B}}$ is proportional to h/R_2.

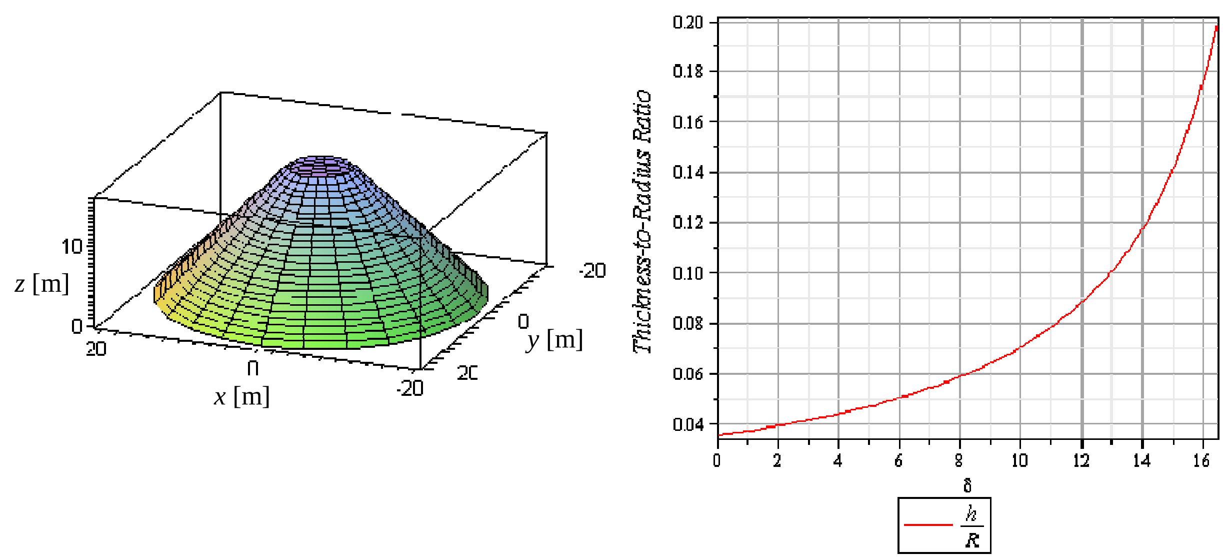


Figure 3. Cone geometry.

For equal thickness and local radii of curvature the effect on the conical shell is larger compared to the case of the paraboloid. This effect is clearly shown by the right-hand side of Fig. 3 and Fig. 4 (note that *h*=2 in section IV-A) and will be discussed afterwards.

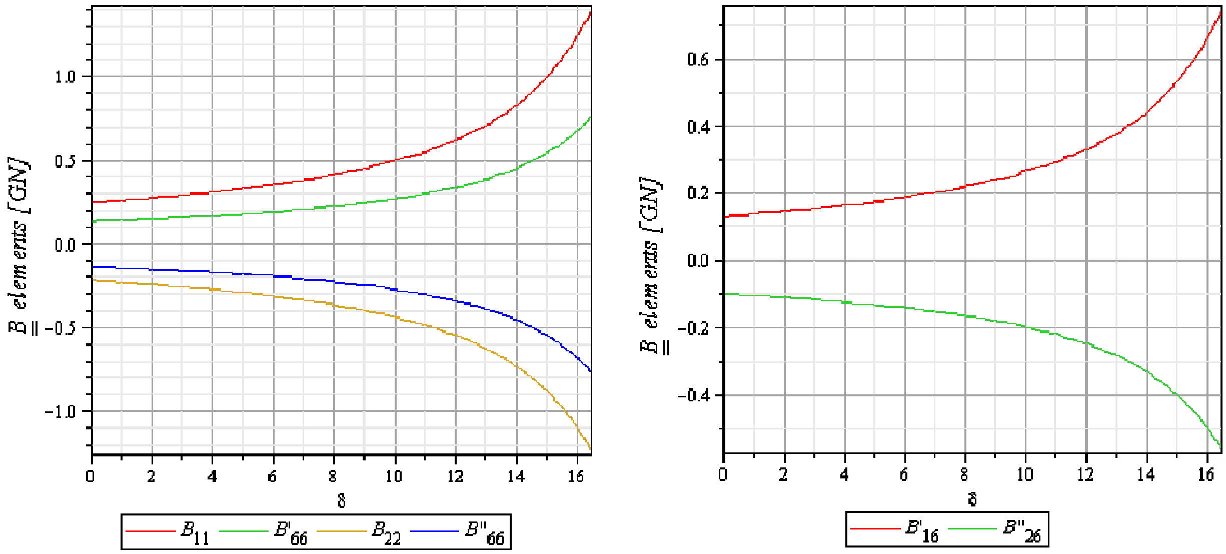


Figure 4. B matrix elements.

E. Hyperbolic Paraboloid ($R_1 R_2 < 0$, h =1m)

For a hyperbolic paraboloid δ_1 and δ_2 represent the orthogonal curvilinear coordinate system described by the grid lying on the surfaces shown in Fig. 5. The expressions for both CLT and new $\underline{\underline{B}}$ are

$$\underline{\underline{B}}\left(R_1(\delta_1,\delta_2),R_2(\delta_1,\delta_2)\right)=\begin{bmatrix} B_{11} & B'_{16} & 0 & 0 \\ B'_{16} & B'_{66} & 0 & 0 \\ 0 & 0 & B_{22} & B''_{26} \\ 0 & 0 & B''_{26} & B''_{66} \end{bmatrix};\quad \underline{\underline{B}}(CLT)=\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{44}$$

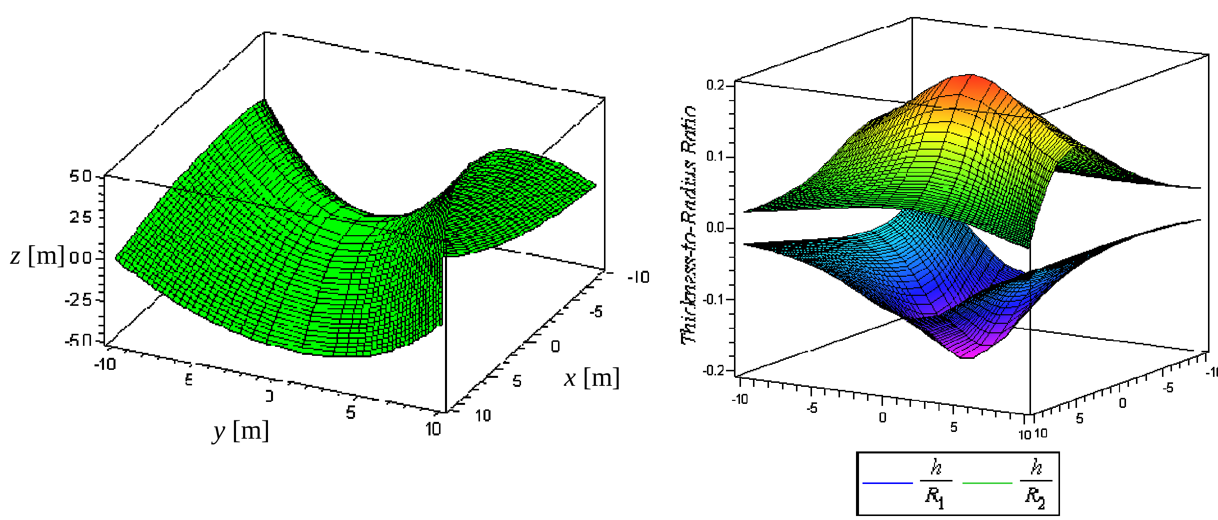


Figure 5. Hyperbolic Paraboloid geometry.

The distribution of new B_{ij} is shown in Fig. 6 and shows significant values with their maximum occurring at the centre of the structure.

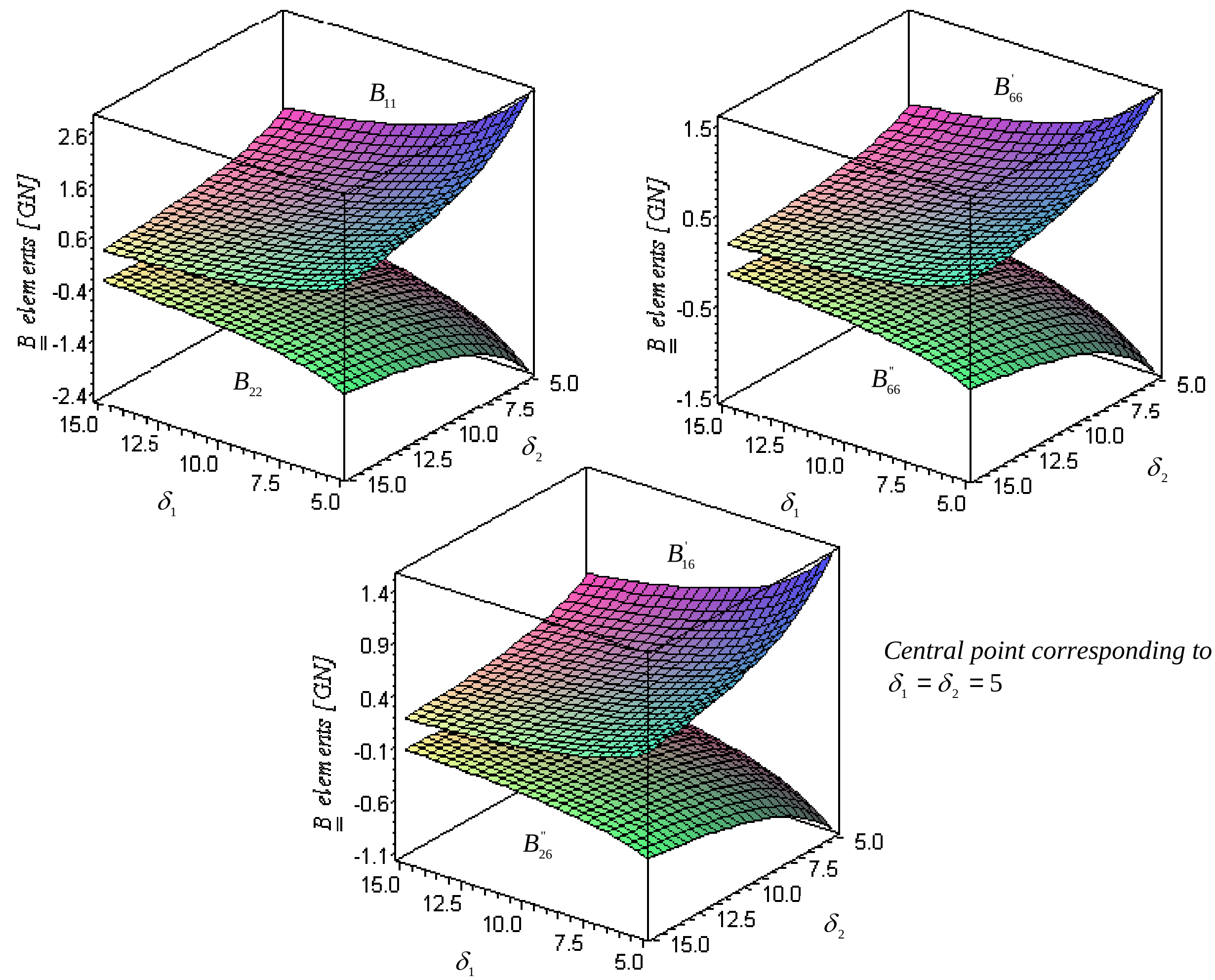


Figure 6. B matrix elements over the positive quadrant of the structure. (Note color contours indicate perspective only.)

V. Discussion of Case Study Results

For all of the case studies considered, with the notable exception of the sphere, the $\underline{\underline{B}}$ matrix is non-zero, even for symmetrically laminated structures, due to the inherent geometry. An important general rule may be deduced from the expressions in Eq. (37), and that is, the effect of the initial geometry on the elastic behavior of a curved surface depends on its Gaussian curvature, G. This quantity is defined as the product of the principal curvatures and it is positive for synclastic surfaces (paraboloid of revolution), zero for developable or ruled surfaces (cylinder, cone) and negative for anticlastic surfaces (hyperbolic paraboloid). For structures with different geometries and identical thicknesses, the magnitude of the elements of $\underline{\underline{B}}$ increases for decreasing G. In summary (from Eq. (37)),

$$G > 0 \quad \Rightarrow \quad B_{ij} \cong O\left(\frac{h^3}{|R_2|} - \frac{h^3}{|R_1|}\right)$$

$$G = 0 \quad \Rightarrow \quad B_{ij} \cong O\left(\frac{h^3}{|R_2|}\right) \tag{45}$$

$$G < 0 \quad \Rightarrow \quad B_{ij} \cong O\left(\frac{h^3}{|R_2|} + \frac{h^3}{|R_1|}\right).$$

Interestingly, the significance of the B_{ij} terms depends only on geometry, via G, and not on material stiffness properties. As such, the effect of curvature on B_{ij} is completely captured by G, noting that largest effects occur for anticlastic geometries, such as hyperbolic paraboloid (negative G) and smallest for synclastic curvatures (positive G) with zero effect for spheres. Curvature effects on B_{ij}, for cylindrical shells, are intermediate between the two previous examples, as may be expected, due to their zero G value.

VI. Overall Discussion

It is noted that the linear part of the developed model is in good agreement with results from Ref. 3; the main difference is the expression of the stiffness matrix coefficients. In Ref. 3, the authors carried out the integration over the thickness required to find these coefficients (see section II-D) numerically approximating certain terms. In the present work, this approximation is avoided and, as such, led to different analytical results. The model has been further extended to include the effects of geometrically nonlinear deformations.

Preliminary numerical analysis show a good degree of consistency with respect to results presented in Ref. 26. Results also show variations of 5-10% on the strain components with respect to their classical values. It is noted that such a difference may significantly affect buckling and post-buckling phenomena. Future work will address this issue.

VII. Conclusion

General equations of multilayered anisotropic shells were developed by including the effects of shear deformation, initial curvature and geometrically nonlinear deformation effects. A novel expression for the stiffness matrix has been presented in which the relationship between the shell shape and the stiffness coefficients has been made explicit. Notably, the part of the laminate constitutive equations describing linear deformations exhibit symmetrical stiffness coefficients, within the novel 4x4 matrix formulation, even though $N_{12} \neq N_{21}$ and $M_{12} \neq M_{21}$. The role of the geometry (initial curvatures) as a source of anisotropy has been then analyzed. It has been shown that the effect of curvature significantly affects $\underline{\underline{B}}$ and that its magnitude depends on the sign of the Gaussian curvature. Generally, each element of the stiffness matrix partially depends on the thickness/local radius of curvature ratio and on the Gaussian curvature.

Acknowledgments

The authors thank the funding of Airbus UK and HERDA-SW through the Great Western Research grant.

References

[1]Reddy, J. N., *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*, 2nd ed., CRC Press, Boca Raton, Florida, 2004.

[2]Novozhilov, V. V., *Foundation of the Nonlinear Theory of Elasticity*, Dover Publications, Inc., Mineola, New York, 1999.

[3]Toorani, M. H., Lakis, A. A., “General Equations of Anisotropic Plates and Shells Including Transverse shear Deformation, Rotary Inertia and Initial Curvature Effects,” *Journal of Sound and Vibration*, Vol. 237, No. 4, 2000, pp. 561, 615.

[4]Calladine, C. R., *Theory of shell Structures*., Cambridge University Press, Cambridge, UK, 1983.

[5]Gol’denveizer, A.L., *Theory of Elastic Thin Shells*, Pergamon Press Ltd., London, UK, 1961.

[6]Novozhilov, V. V., *The Theory of Thin Shells*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1959.

[7]Hoskin, B. C., Baker, A. A., *Composite Materials for Aircraft Structures*, American Institute of Aeronautics and Astronautics, Inc., New York, 1986.

[8]Pao, Y. C., “Simple Bending Analysis of Laminated Plates by Large-Deflection Theory,” *Journal of Composite Materials* Vol. 4, pp. 380-389, 1970.

[9]Donnell, L. H., *Beams, Plates and Shells,* McGraw-Hill, New York, 1976.

[10]Niordson, F. I., *Shell Theory*, North-Holland series in applied mathematics and mechanics, v. 29, Amsterdam, North-Holland, 1985.
[11]Palazotto, A. N., Dennis, S. T., *Nonlinear Analysis of Shell Structures*, AIAA education series, American Institute of Aeronautics and Astronautics, Washington, DC, 1992.
[12]Ogden, R. W., *Non-Linear Elastic Deformations*, Dover Publications, Inc., Mineola, New York, 1984.
[13]Pogorelov, A. V., *Bendings of Surfaces and Stability of Shells*, Translations of mathematical monographs, v. 72, American Mathematical Society, Providence, Rhode Island, 1988.
[14]Chia, C.-Y., *Nonlinear Analysis of Plate*, McGraw-Hill International Book Co, New York, 1980.
[15]Cox, H. L., *The Buckling of Plates and Shells*, Pergamon Press, Oxford, 1963.
[16]Sanders Jr., J. L., “Nonlinear theories for thin shells,” *Quarterly of Applied Mathematics*, Vol. 21, No. 1, 1963, pp. 21, 36.
[17]Sanders Jr., J. L., “An Improved First Approximation theory for Thin Shells,” NASA TR-R24, 1959.
[18]Widera., O. E., “An Asymptotic theory for the Moderately Large deflections of Anisotropic Plates,” *Journal of Engineering Mathematics*, Vol. 3, No. 3, 1969, pp. 239, 244.
[19]Librescu, L., Schmidt, R., “Refined Theories of Elastic Anisotropic Shells Accounting for Small Strains and Moderate Rotations,” *International Journal of Non-Linear Mechanics*, Vol. 23, No. 3, 1988, pp. 217, 229.
[20]Pai, P. F. and Nayfeh, A. H., “A nonlinear composite plate theory,” *Nonlinear Dynamics*, Vol. 3, No. 6, 1992, pp. 431, 463.
[21]Pai, P. F. and Nayfeh, A. H., “A unified nonlinear formulation for plate and shell theories,” *Nonlinear Dynamics*, Vol. 6, No. 4, 1994, pp. 459, 500.
[22]Yu, W., Hodges, D. H., “A Geometrically Nonlinear Shear Deformation Theory for Composite Shells,” *Journal of Applied Mechanics*, Vol. 71, No. 1, 2004, pp. 1, 9.
[23]Cho, M., and Parmerter, R. R., “Efficient Higher-Order Composite Plate Theory for General Lamination Configurations,” *AIAA Journal*, Vol. 31, 1993, pp. 1299, 1306.
[24]Chia, C.-Y., Chia, D. S., “Effect of transverse shear on finite deformation of generally laminated shallow shell with double curvature,” *Acta Mechanica Solida Sinica*, Vol. 9, No. 1, 1996, pp. 27, 41.
[25]Chia, C.-Y., “Nonlinear analysis of doubly curved symmetrically laminated shallow shells with rectangular planform,” *Archive of Applied Mechanics (Ingenieur Archiv)*, Vol. 58, No. 4, 1988, pp. 252, 264.
[26]Weaver, P. M., “On Beneficial Anisotropic Effects in Composite Structures,” *43rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, Denver, Colorado, Apr. 22-25, 2002.